Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 23, 2004*
Accepted: *September 29, 2004*

Supersymmetric null-surfaces

Andrei Mikhailov

California Institute of Technology 452-48
Pasadena CA 91125, U.S.A., and
Institute for Theoretical and Experimental Physics
117259, Bol. Cheremushkinskaya, 25, Moscow, Russia
E-mail: andrei@theory.caltech.edu

Abstract: Single trace operators with the large R-charge in supersymmetric Yang-Mills theory correspond to the null-surfaces in $AdS_5 \times S^5$. We argue that the moduli space of the null-surfaces is the space of contours in the super-grassmanian parametrizing the complex (2|2)-dimensional subspaces of the complex (4|4)-dimensional space. The odd coordinates on this super-grassmanian correspond to the fermionic degrees of freedom of the superstring.

Keywords: Superstrings and Heterotic Strings, AdS-CFT and dS-CFT Correspondence, Superspaces, Differential and Algebraic Geometry.

JHEP09(2004)068

© SISSA/ISAS 2004

Contents

JHEP09(2004)068

1. Introduction

In the AdS/CFT correspondence the single-trace operators of the large N supersymmetric Yang-Mills theory correspond to the single string states in the Type IIB theory on $AdS_5 \times S^5$. The single-trace operators have a form of the trace of the product of L elementary fields of the Yang-Mills theory. There is a certain subclass of these operators for which the corresponding string can be described semiclassically [1]–[4]. For an operator to correspond to a classical dual string, one has to take $L \to \infty$. Also, in taking this limit, one has to arrange the elementary fields under the trace in such a way that the operator is "locally half-BPS" [5].

Let us explain what "locally half-BPS" means. The $N = 4$ theory has six scalar fields $\Phi_1 \ldots, \Phi_6$. Let us consider a complex combination $Z = \Phi_5 + i\Phi_6$. In the limit $L \to \infty$ we have to require that each elementary field under the trace is of the form $\phi_k = g_k.Z$ where g_k is some element of the superconformal group $\mathrm{SU}(2,2|4)$. In other words, each elementary field is in the superconformal orbit of Z. Moreover we should have $g_{k+1} = g_k + O(1/L)$. Therefore instead of the discreet "chain" of the group elements $g_1, \ldots, g_L$ we have a continuous contour in the group manifold $g(\sigma)$, where $\sigma = 2\pi k/L$. In this "continuous limit" the anomalous dimension of the Yang-Mills operator becomes of the order λ/L plus the higher order corrections which are the series in $\frac{\lambda}{L^2}$. Moreover, the renormgroup flow defines a classical dynamical system on the space of contours $g(\sigma)$ (see [6] and references therein). More precisely, $g(\sigma)$ takes values not in the group manifold $\mathrm{PSU}(2,2|4)$ itself but rather in the coset space which is $\mathrm{PSU}(2,2|4)$ modulo the subgroup which acts on Z as a phase rotation. Therefore the renormgroup flow in the field theory defines a classical dynamical system on the space of loops $g(\sigma)$ taking values in the supercoset $\mathrm{PSU}(2,2|4)/(\mathrm{PSU}(2|2)\times\mathrm{PSU}(2|2)\times\mathrm{U}(1)^2)$.

This supercoset has dimension $(16|16)$, sixteen even and sixteen odd coordinates. Therefore the "continuous" Yang-Mills operators are described by 16 even and 16 odd functions of one real variable. But 16 even and 16 odd functions also parametrize the phase space of the classical Type IIB superstring in $AdS_5\times S^5$. Therefore, it is natural to conjecture that the classical dynamical system on the space of the locally half-BPS operators defined by the renormgroup flow is equivalent to the classical worldsheet dynamics of the Type IIB superstring.

Unfortunately we do not know any independent prescription which would tell us which string worldsheet corresponds to a given Yang-Mills operator. But the conjecture that the string sigma-model is equivalent to the classical renormgroup flow is nontrivial even without such a prescription. Indeed, the equivalence of two dynamical systems is already a nontrivial statement. It was partially verified at the two loop level (in three different ways!) in [8]–[10].

Let us briefly review what happens at the one loop level, following [11, 12] (see also [13, 14] for a different approach). Consider the string worldsheet corresponding to a given Yang-Mills single trace operator composed of L elementary fields. The shape of the worldsheet depends on the coupling constant λ. When $\frac{\lambda}{L^2}\to 0$ the worldsheet degenerates and becomes a null-surface. Moreover, this null-surface comes with a parametrization of the light rays. Therefore in the "continuous" limit the single trace operators correspond to the parametrized null surfaces. It turns out that the string worldsheet theory defines the structure of a hamiltonian system on the space of parametrized null-surfaces. The definition of this hamiltonian system goes as follows. Pick a parametrized null-surface. Consider a family of extremal surfaces depending on the parameter ϵ, such that: 1) the limit when $\epsilon\to 0$ is our null-surface and 2) the density of the conserved charges on the worldsheet in the limit $\epsilon\to 0$ is proportional to $\frac{1}{\epsilon}d\sigma$ where σ is the parametrizing function (see [12, section 3.3] for the precise formula). There are infinitely many such families, but for the purpose of our definition we can pick any one, satisfying these two properties. The deviation of this extremal surface from our null-surface is locally of the order ϵ^2. But if we follow its evolution in the global AdS time, the deviation will accumulate. After the time interval of the order $\Delta T\sim\frac{1}{\epsilon^2}$, the deviation becomes of the order one, and our extremal surface will locally approximate (with the accuracy $\sim\epsilon^2$) another null-surface. This determines the evolution on the space of null-surfaces.

An important point is that this definition does not depend on the choice of the family converging to the null-surface. If we pick two different worldsheets approximating the same parametrized null-surface, they will "oscillate" around each other, but the deviation between them will not accumulate in time. Therefore, different approximating surfaces determine the same "slow evolution" on the space of null-surfaces.

The space of parametrized null-surfaces is identified with the space of pairs of functions $Y:\ S^1\to\mathbb{C}^{2+4}$, $Z:\ S^1\to\mathbb{C}^6$ such that $|Y(\sigma)|^2=|Z(\sigma)|^2=2$ and $(Y(\sigma),Y(\sigma))=(Z(\sigma),Z(\sigma))=0$ and $(\overline{Y},\partial_\sigma Y)=(\overline{Z},\partial_\sigma Z)$. The one-loop anomalous dimension corre-

JHEP09(2004)068

sponds to the hamiltonian of the slow evolution:

$$\Delta = \frac{1}{16\pi^2}\frac{\lambda}{(L/2\pi)}\int_0^{2\pi} d\sigma\left((\partial_\sigma \overline{Z}, \partial_\sigma Z) - (\partial_\sigma \overline{Y}, \partial_\sigma Y)\right). \tag{1.1}$$

This space is a $U(1)$-bundle over a submanifold of the loop space of the product of two grassmanians:

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2)\times \mathrm{SO}(4)} \times \frac{\mathrm{SO}(6)}{\mathrm{SO}(2)\times \mathrm{SO}(4)}\,. \tag{1.2}$$

It consists of the loops satisfying certain integrality condition which corresponds to the cyclic invariance of the trace.

In this paper we will argue that the fermionic degrees of freedom on the worldsheet parametrize in the ultrarelativistic limit the odd directions of the supercoset space. Just as the fast-moving bosonic string corresponds to a contour in the product of two grassmanians (1.2) the superstring defines a contour in

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{PSU}(2|2)\times \mathrm{PSU}(2|2)\times \mathrm{U}(1)^2}\,.$$

Turning on the fermionic degrees of freedom of the null-surface corresponds on the field theory side to considering operators with insertions of the fermions and the field strength.

Dynamical systems on supersymmetric coset spaces were studied in the recent papers [15].

2. Single trace operators with large spin

The single trace operators are the operators of the form $\mathrm{tr}\,\phi_1\phi_2\cdots\phi_n$ where $\phi_1,\ldots,\phi_n$ are the fundamental fields. The one loop anomalous dimension for all the single trace operators in the $\mathcal{N}=4$ Yang-Mills theory was computed in [16, 17]. For the one loop computation, each fundamental field under the trace can be considered a free field, and therefore it transforms in the singleton representation of the superconformal group $\mathrm{SU}(2,2|4)$. The one loop anomalous dimension corresponds to the hermitean operator (interaction hamiltonian) acting on the trace of the product of the free fields. It was shown in [17] that this operator commutes with the generators of the superconformal group. In the planar limit the interaction is a sum of the pairwise interactions of all the fundamental fields under the trace which are next to each other in the product. In other words, the one loop anomalous dimension of the operator $\mathrm{tr}\,\phi_1\phi_2\cdots\phi_n$ is given by the sum of the diagramms involving ϕ_1 and ϕ_2, diagramms involving ϕ_2 and ϕ_3 and so on. The one loop interaction preserves the number of the fundamental fields under the trace.

2.1 Coherent states

The "continuous" approach to the computation of the anomalous dimension of the single trace operator was proposed in [6]. This approximation is useful when the number of the fields under the trace is very large. This approach (as we understand it) relies on the existence of the special set of vectors in the singleton representation. This set of vectors

JHEP09(2004)068

is obtained in the following way. Consider the free $\mathcal{N}=4$ theory on $\mathbb{R}\times S^3$. The vacuum is conformally invariant. Let us act on the vacuum by the creation operator of the free boson $Z(x)=\phi_5(x)+i\phi_6(x)$ integrated over S^3. We will get a state which we will call $\psi_{\mathbf{1}}$. This state is not invariant under SU(2, 2|4). For any group element $g\in \mathrm{SU}(2,2|4)$ we will denote $\psi_g = g.\psi_{\mathbf{1}}$. The states of this form are parametrized by the points of the coset space

$$Gr(2|2,4|4)=\frac{\mathrm{PSU}(2,2|4)}{\mathrm{PSU}(2|2)\times\mathrm{PSU}(2|2)\times\mathrm{U}(1)^2} \tag{2.1}$$

because $\mathrm{PSU}(2|2)\times\mathrm{PSU}(2|2)\times\mathrm{U}(1)^2$ is the subgroup which acts on $\psi_{\mathbf{1}}$ as a phase rotation. These states generate the singleton representation. It is then conjectured that in the limit of the large number of fields the "semiclassical" states are the decomposable tensors of the form

$$\psi_{g_1}\otimes\psi_{g_2}\otimes\cdots\otimes\psi_{g_n}\,, \tag{2.2}$$

where the difference between g_k and g_{k+1} is of the order $\frac{1}{n}$. In the continuum limit $n\to\infty$, the number of the site k becomes a continuous parameter $\sigma = k/n$, and the evolution of the state is approximated by the classical evolution of the contour $g(\sigma)$. The classical hamiltonian is the matrix element

$$H_{cl}[g(\sigma)] = (\overline{\psi}_{g_1}\otimes\overline{\psi}_{g_2}\otimes\cdots\otimes\overline{\psi}_{g_n},\ H_{int}\ \ \psi_{g_1}\otimes\psi_{g_2}\otimes\cdots\otimes\psi_{g_n})\,. \tag{2.3}$$

The symplectic structure is $\Omega=\int d\sigma\ \Omega(\sigma)$ where $\Omega(\sigma)$ is the differential of the one-form

$$\alpha(\sigma)=\left(\overline{\psi}_{g(\sigma)}, d\psi_{g(\sigma)}\right). \tag{2.4}$$

This data defines a dynamical system on the space of contours in the super-grassmanian (2.1). The interaction hamiltonian of [17] involves only the pairs of neighbors in the product. Therefore the continuous hamiltonian should be a local functional of the contour. Moreover, one can see that it contains not more than two derivatives ∂_σ. Therefore the value of the hamiltonian on the contour should be given by the value of the classical action of the free particle on the super-grassmanian which has this contour as a trajectory.[1]

It is useful to write down these coherent states more explicitly. The space of one-particle states of the free scalar field theory can be identified with the space of the positive-frequency solutions of the free field equations. We will describe a family of positive-frequency solutions parametrized by the points of the coset $\frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2)\times\mathrm{SO}(4)}\times\frac{\mathrm{SO}(6)}{\mathrm{SO}(2)\times\mathrm{SO}(4)}$. This coset is parametrized by two complex lightlike vectors Y and Z, $(\overline{Y},Y)=(\overline{Z},Z)=2$, $(Y,Y)=(Z,Z)=0$, modulo independent phase rotations of Y and Z. The manifold of the complex lightlike vectors Y in $\mathbb{C}^{2+4}$ consists of two connected components. Those Y which can be rotated by SO(2, 4) to $(1,-i,0,0,0,0)$ belong to the first component, and those which can be rotated to $(1,i,0,0,0,0)$ belong to the second component. We need only the first component. Given Y and Z, let us consider the positive frequency wave of

[1]The contour can be an arbitrary trajectory, not necessarily satisfying the equations of motion.

JHEP09(2004)068

the field $Z_1\Phi_1 + \cdots + Z_6\Phi_6$ parametrized by Y:

$$
\begin{aligned}
f_{[Y]}(\tau,\mathbf{n}) &= \frac{1}{Y_{-1}\cos\tau + Y_0\sin\tau - (\mathbf{Y},\mathbf{n})} \\
\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\mathbf{n}^2}\right) f_{[Y]} &= -f_{[Y]}\,.
\end{aligned}
\tag{2.5}
$$

Here $(\tau,\mathbf{n})$ are the coordinates on $\mathbb{R}\times S^3$, τ is parametrizing $\mathbb{R}$ and $\mathbf{n} = (n_1, n_2, n_3, n_4)$ is the unit vector parametrizing S^3. This defines a state $\psi_{[Y],[Z]}$.

The super-grassmanian (2.1) is the (4,2,2) analytic superspace of [18]–[20]. The bosonic coset space $\frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2)\times\mathrm{SO}(4)} \simeq \frac{\mathrm{SU}(2,2)}{S(U(2)\times U(2))}$ was discussed in the context of the AdS/CFT correspondence in [21], where it was identified as the moduli space of timelike geodesics in AdS_5. This coset space is the future tube of the Minkowski spacetime (see [21] and references therein, and [22, 23] for the mathematical background). The motion of a particle in AdS space was also studied in [24]. An interesting technique for obtaining the singleton representation from dynamics in higher dimensional spaces was developed in [25]; in the next section we will use an approach related to the ideas of [25] to study spinors in AdS space. The general theory of the coherent states of the type discussed in this section was developed in [22].

The singleton representation of $\mathrm{SO}(2,4)$ is not square integrable (see [26] for a recent discussion of this fact, and references therein). Therefore our system of coherent states does not resolve the identity.

2.2 Anomalous dimension

The coherent states of the spin chain generally speaking do not have a definite energy in the $\lambda = 0$ theory. In fact the corresponding operators are superpositions of operators with different engineering dimensions. In this situation we can define the one-loop anomalous dimension in the following way. The superconformal group is actually $\widetilde{PSU}(2,2|4)$ — the universal covering of $\mathrm{PSU}(2,2|4)$. The universal covering has a center $C = \mathbb{Z}$. Let c denotes the generator of the center. In the free field theory $c = 1$, but it acts nontrivially in the interacting theory:

$$c = e^{i\Delta}\,. \tag{2.6}$$

In perturbation theory Δ is expanded in powers of λ. It starts with the term linear in λ, which is the one-loop anomalous dimension. It is obvious from this definition that the one-loop anomalous dimension commutes with the superconformal group.

At the one-loop level the action of the center is the sum of the contributions of the pairwise interactions of the nearest neighbors of the parton chain. Each pairwise interaction separately commutes with the superconformal group. Therefore it is enough to consider the case when the pair of nearest neighbors is:

$$\cdots \otimes Z(0) \otimes (Z(0) + a\mathcal{O}_1 + a^2\mathcal{O}_2 + \cdots) \otimes \cdots\,. \tag{2.7}$$

Here a is the lattice spacing and $\mathcal{O}_n$ are some combinations of elementary fields. For the continuous operators

$$\mathcal{O}_1 = \alpha_I \Phi^I(0) + \beta^\mu \partial_\mu Z(0)\,, \tag{2.8}$$

JHEP09(2004)068

where α_I ($I = 1, 2, 3, 4$) and β^μ are some complex coefficients. We need to compute the expectation value of the interaction hamiltonian in the coherent state:

$$\langle H \rangle = \left(\overline{Z}(0) \otimes (\overline{Z}(0) + a\overline{\mathcal{O}}_1 + \cdots), H \ Z(0) \otimes (Z(0) + a\mathcal{O}_1 + \cdots) \right) .$$

Notice that $H.Z(0) \otimes Z(0) = 0$ because $Z(0) \otimes Z(0)$ is the vacuum ("the BMN vacuum"). Therefore

$$\langle H \rangle = a^2 (\overline{Z}(0) \otimes \overline{\mathcal{O}}_1, \ H \ \ Z(0) \otimes \mathcal{O}_1) + o(a^2) . \tag{2.9}$$

All that we need in the continuum limit is the leading term. The action of H was computed in [17]. We will now briefly review some results of [17]. The tensor product of two supersingletons V_F of PSU$(2, 2|4)$ is a reducible representation. It decomposes into the direct sum of irreducible representations of PSU$(2, 2|4)$:

$$V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j \ . \tag{2.10}$$

The action of H is the sum of the projectors P_j on V_j, with the coefficients depending on j. The coefficient of P_0 is zero, and the coefficient of P_1 is $\frac{\lambda}{4\pi^2}$. The BMN vacuum $Z(0) \otimes Z(0)$ belongs to V_0. What can we say about $Z(0) \otimes \mathcal{O}_1$ where $\mathcal{O}_1$ is given by (2.8)? One can see that the symmetric part $Z(0) \otimes \mathcal{O}_1 + \mathcal{O}_1 \otimes Z(0)$ belongs to V_0 and the antisymmetric part $Z(0) \otimes \mathcal{O}_1 - \mathcal{O}_1 \otimes Z(0)$ belongs to V_1. This means that:

$$(\overline{\Psi}, H.\Psi) = (\overline{\Psi}, \frac{\lambda}{8\pi^2} \sum_{l=1}^{L} (1 - P_{l,l+1})\Psi) + \cdots , \tag{2.11}$$

where dots denote terms subleading in the continuous limit. To compute the right hand side, we need the scalar product $(\overline{\psi}_{[Y],[Z]}, \psi_{[Y'],[Z']}) = \frac{(\overline{Z}, Z')}{(\overline{Y}, Y')}$. Therefore the one-form (2.4) is $(\overline{Y}, dY) - (\overline{Z}, dZ)$, and the classical hamiltonian $(\overline{\Psi}, H.\Psi)$ is given by (1.1).

3. Fast moving superstrings

3.1 Anomalous dimension as a deck transformation

AdS space is the universal covering space of the hyperboloid. The center of $\widetilde{PSU}(2, 2|4)$ acts as a deck transformation exchanging the sheets. We can visualize the action of this deck transformation on the string phase space in the following way. Let us replace AdS_5, which is the covering space of the hyperboloid, by the hyperboloid itself $AdS_5/\mathbb{Z}$. Let us formally consider the string as living on $(AdS_5/\mathbb{Z}) \times S^5$. Let us pick a point x on the string worldsheet Σ. Consider a neighborhood of x in $(AdS_5/\mathbb{Z}) \times S^5$ which is simply connected. For example, we can pick as such a neighborhood a set of points which are within the distance $R/2$ from x, where R is the radius of AdS_5. Let B denote such a neighborhood. Consider the part of the string worldsheet which is inside B (that is, $B \cap \Sigma$). One can see that $B \cap \Sigma$ consists of several sheets, which can be enumerated. These sheets are two-dimensional, so we can think of them as cards; $B \cap \Sigma$ is then a deck of cards. Let x belong to the sheet number n, then we can draw a path on Σ starting at x, winding once on

JHEP09(2004)068

the noncontractible cycle in $AdS_5/\mathbb{Z}$ and then ending on the sheet number $n+1$. Let Σ_n denote the sheet number n. The coordinate distance between Σ_n and Σ_{n+1} is of the order ϵ^2. The deck transformation maps Σ_n to Σ_{n+1}, Σ_{n+1} to Σ_{n+2} and so on. This determines the action of the discreet group $\mathbb{Z}$ on the string phase space, which should be identified with the action of the center of the superconformal group on the field theory side.

In this paper we consider only the one-loop approximation. Therefore we need the action of $\mathbf{Z}$ to the order ϵ^2. At this order the deck transformation can be interpreted as the slow evolution. Indeed, let us replace n with the continuous parameter $t = n\epsilon^2$. For every n the corresponding sheet Σ_n is close to some null-surface which we denote $\Sigma(0)^{(t)}$; this null-surface is defined for each t; there is an ambiguity in the definition of $\Sigma(0)^{(t)}$ but it is of the order ϵ^2. Therefore, in the limit $\epsilon^2 \to 0$ the deck transformations define a one-parameter family of transformations of the null-surfaces. This slow evolution of the null surfaces was studied in [12] but only in the bosonic sector. In this section we will turn on the fermionic degrees of freedom.

3.2 Supersymmetric null surfaces

The worldsheet theory for the superstring in $AdS_5 \times S^5$ can be formulated as a sigma-model with the target space the supercoset

$$M = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}\,. \tag{3.1}$$

In the ultrarelativistic limit the string worldsheet becomes a parametrized null-surface. The parametrized null-surfaces correspond to the loops in the coset space

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2)\times\mathrm{SO}(4)}\times\frac{\mathrm{SO}(6)}{\mathrm{SO}(2)\times\mathrm{SO}(4)}\simeq\frac{\mathrm{SU}(2,2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))}\times\frac{\mathrm{SU}(4)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2))}\,.$$

This is the bosonic part of the story. Let us study the effect of the fermionic degrees of freedom.

We will apply the Green-Schwarz approach for the string in $AdS_5\times S^5$ developed in [27], but using a different representation of the gamma-matrices. Following [28] we will represent the spinors in the space $AdS_5 \times S^5$ as restrictions of spinors in the flat space $\mathbb{R}^{2+10}$. We prefer this representation, because the covariantly constant spinors on $AdS_5\times S^5$ correspond in this picture to the constant spinors in the flat space. This also agrees with the philosophy of [25]. Consider the embeddings $AdS_5 \subset \mathbb{R}^{2+4}$ with coordinates $X_{-1}, X_0, \ldots, X_4$ and $S^5 \subset \mathbb{R}^6$ with coordinates $X_5, \ldots, X_{10}$. Consider the twelve-dimensional chiral spinors Ψ of $\mathrm{SO}(2,10)$. These twelve-dimensional spinors are sections of the spinor bundle over $\mathbb{R}^{2+10}$, which is a trivial bundle (the product $\mathbb{R}^{2+10}\times\mathbb{C}^{32}$). Let us restrict this bundle to $AdS_5\times S^5 \subset \mathbb{R}^{2+10}$. This restriction would be a trivial bundle $AdS_5\times S^5\times\mathbb{C}^{32}$. It turns out that the spinor bundle on $AdS_5 \times S^5$ can be realized as a subbundle of this bundle. This subbundle is the image of the projector $\frac{1}{2}(1+\Gamma^A\Gamma^S)$ where Γ^A is the Γ-matrix corresponding to the vector of the length square -1 orthogonal to AdS_5 in $\mathbb{R}^{2+4}$ and Γ^S corresponds to the unit vector orthogonal to S^5 in $\mathbb{R}^6$. Therefore a section of the

JHEP09(2004)068

spinor bundle on $AdS_5 \times S^5$ can always be represented in the form:

$$\psi = \frac{1}{2}\left(1 + \Gamma^A \Gamma^S\right) \Psi_{++}\,, \tag{3.2}$$

where Ψ_{++} satisfies:

$$\begin{aligned} \Gamma_{-1}\Gamma_0 \cdots \Gamma_4 \Psi_{++} &= i\Psi_{++} \\ \Gamma_5 \cdots \Gamma_{10} \Psi_{++} &= i\Psi_{++}\,. \end{aligned} \tag{3.3}$$

This condition means that Ψ_{++} is the product of the positive chirality spinor of SO(2,4) and the positive chirality spinor of SO(6). We will choose the Γ-matrices to be real. We will denote $\rho_{\mathrm{SU}(2,2)}$ the space of the positive chirality spinor representation of SO(2,4) (the fundamental of SU(2,2)) and $\rho_{\mathrm{SU}(4)}$ the space of the positive chirality spinor representation of SO(6) (the fundamental of SU(4)). Therefore $\Psi_{++} \in \rho_{\mathrm{SU}(2,2)} \otimes \rho_{\mathrm{SU}(4)}$.

The worldsheet degrees of freedom are two Majorana-Weyl spinors $\theta^1(\tau,\sigma)$, $\theta^2(\tau,\sigma)$. We will parametrize them by a single complex Ψ_{++}:

$$\begin{aligned} \theta^1 &= (1 + \Gamma^A\Gamma^S)\mathrm{Re}(\Psi_{++}) \\ \theta^2 &= (1 + \Gamma^A\Gamma^S)\mathrm{Im}(\Psi_{++})\,. \end{aligned} \tag{3.4}$$

The γ-matrices of [27] are related to the gamma-matrices in the tangent space to $AdS_5 \times S^5$:

$$i\gamma^a = \Gamma^a \hat{F}_A\,, \qquad \gamma^{a'} = \Gamma^{a'}\hat{F}_S\,, \qquad \hat{F}_A\theta^I = \hat{F}_S\theta^I\,. \tag{3.5}$$

Here $\hat{F}_A$ is the product of the five gamma-matrices Γ_a tangent to AdS_5 and $\hat{F}_S$ is the product of gamma-matrices tangent to S^5. Also, for any vector v in the tangent space to $AdS_5 \times S^5$ we will denote:

$$\hat{v} = \Gamma_\mu v^\mu\,.$$

Let us introduce a notation:

$$\overline{\theta} = \theta^T \Gamma_{-1}\Gamma_0\,, \tag{3.6}$$

where the superindex T denotes the transposition. In this definition we assume that we have chosen the gamma-matrices so that Γ_{-1} and Γ_0 are antisymmetric, and $\Gamma_1, \ldots, \Gamma_{10}$ are symmetric. The definition (3.6) is for a real spinor θ. For a complex combination $\theta_1 + i\theta_2$ we define

$$\overline{\theta_1 + i\theta_2} = \overline{\theta}_1 - i\overline{\theta}_2\,. \tag{3.7}$$

This notation allows us to write an explicit formula for the linear map from the tensor product of two chiral spinor bundles to the vector bundle:

$$\theta_1 \otimes \theta_2 \mapsto j\,, \qquad j^\mu = \overline{\theta}_1 \Gamma^A . \Gamma^\mu \theta_2 \tag{3.8}$$

We will use the same notation for the conjugate of Ψ_{++}:

$$\overline{\Psi_{++}} = \Psi_{++}^{*T}\Gamma_{-1}\Gamma_0\,, \tag{3.9}$$

where Ψ^*_{++} means the complex conjugate of Ψ_{++} (notice that Ψ_{++} has to be a complex spinor).

JHEP09(2004)068

The covariant derivative modified by the Ramond-Ramond field strength is:

$$\mathcal{D}_i(\theta^1 + i\theta^2) = \left[D_i + \frac{1}{4}i(\hat{F}_A - \hat{F}_S)\Gamma_i\right](\theta^1 + i\theta^2)\,. \tag{3.10}$$

The main advantage of considering ten-dimensional spinors as restrictions of twelve-dimensional spinors is a simple form of the covariant derivative:

$$\mathcal{D}_i\left[(1 + \Gamma^A\Gamma^S)\Psi_{++}\right] = (1 + \Gamma^A\Gamma^S)\partial_i\Psi_{++}\,. \tag{3.11}$$

This means that covariantly constant spinors correspond to constant Ψ_{++}.

In this paper we will restrict ourselves to the study of the configurations near $\theta^I = 0$; we will only keep the terms of the lowest order in θ^I (terms quadratic in θ^I in the action). With this restriction, the kappa-transformations are:

$$\delta_k\theta^1 = \widehat{\partial_+ x}\; k^1 \tag{3.12}$$
$$\delta_k\theta^2 = \widehat{\partial_- x}\; k^2 \tag{3.13}$$

and the equations of motion for fermions are:

$$\widehat{\partial_+ x}\;\mathcal{D}_-\theta^1 = 0 \tag{3.14}$$
$$\widehat{\partial_- x}\;\mathcal{D}_+\theta^2 = 0\,. \tag{3.15}$$

They imply that there are spinors η^1, η^2 such that

$$\mathcal{D}_-\theta^1 = \widehat{\partial_+ x}\;\eta^1 \tag{3.16}$$
$$\mathcal{D}_+\theta^2 = \widehat{\partial_- x}\;\eta^2\,. \tag{3.17}$$

Doing the kappa-transformation with the parameters k^1, k^2 such that $D_-k^1 = -\eta^1$ and $D_+k^2 = -\eta^2$ we are left with

$$\mathcal{D}_-\theta^1 = \mathcal{D}_+\theta^2 = 0\,. \tag{3.18}$$

There are some "residual" kappa-transformations which preserve this condition.

For the fast moving string, we choose the coordinates τ, σ so that $g_{\tau\tau} = \epsilon^2$, $g_{\sigma\sigma} = -1$, $g_{\tau\sigma} = 0$. The equations of motion in the "complex" notations is:

$$\mathcal{D}_\tau(\theta^1 + i\theta^2) - \epsilon\left(\mathcal{D}_\sigma(\theta^1 + i\theta^2)\right)^* = 0\,. \tag{3.19}$$

In terms of Ψ_{++}:

$$\partial_\tau\Psi_{++} - \epsilon\,\Gamma^A\Gamma^S(\partial_\sigma\Psi_{++})^* = 0\,. \tag{3.20}$$

What happens when $\epsilon \to 0$? We have $\partial_\tau\Psi_{++} = 0$, thus Ψ_{++} is constant on the light rays forming the null surface. Let us study the residual kappa-transformations. Let $\partial_\tau x_A$ be the AdS-component of the tangent vector to the null-geodesic, and $\partial_\tau x_S$ be the component in the tangent space to the sphere. The following kappa-transformation with the constant parameter K_{++} leaves Ψ_{++} constant along the light rays:

$$\begin{aligned}\delta_K\left[(1 + \Gamma^A\Gamma^S)\Psi_{++}\right] &= \left(\widehat{\partial_\tau x_A} + \widehat{\partial_\tau x_S}\right)(1 + \Gamma^A\Gamma^S)\Gamma^A K_{++} \\ &= (1 + \Gamma^A\Gamma^S)\left(\widehat{\partial_\tau x_A}\Gamma^A + \widehat{\partial_\tau x_S}\Gamma^S\right)K_{++}\,.\end{aligned}$$

JHEP09(2004)068

In other words,

$$\delta_K \Psi_{++} = \left(\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S \right) K_{++} \,. \tag{3.21}$$

The right hand side is constant on the light ray, because $\widehat{\partial_\tau x_A}\Gamma^A$ is the rotation in the equatorial plane of AdS_5 and $\widehat{\partial_\tau x_S}\Gamma^S$ is the rotation in the equatorial plane of S^5. These generators of rotations are very useful. In the spinor language the equator of AdS_5 corresponds to the 2-plane $L_A \subset \rho_{\mathrm{SU}(2,2)}$ which is defined as the subspace on which the rotation in the equatorial plane acts with the eigenvalue $+i$. And the equatorial plane of S^5 corresponds to the 2-plane $L_S \subset \rho_{\mathrm{SU}(4)}$. We have a decomposition:

$$\rho_{\mathrm{SU}(2,2)} \otimes \rho_{\mathrm{SU}(4)} = (L_A \otimes L_S) \oplus (L_A^\perp \otimes S_S^\perp) \oplus (L_A \otimes S_S^\perp) \oplus (L_A^\perp \otimes L_S) \,. \tag{3.22}$$

The kappa-transformation (3.21) then implies that the component of Ψ_{++} which belongs to $(L_A \otimes L_S) \oplus (L_A^\perp \otimes L_S^\perp)$ is a pure gauge; it can be gauged away. This means that we can choose Ψ_{++} to satisfy

$$\left(\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S \right) \Psi_{++} = 0 \tag{3.23}$$

or equivalently $(\widehat{\partial_\tau x_A} - \widehat{\partial_\tau x_S})\theta^I = 0$. In other words Ψ_{++} can be brought into the form

$$\Psi_{++} = \widetilde{\eta} \otimes \phi + \eta \otimes \widetilde{\phi} \,, \tag{3.24}$$

where $\phi, \widetilde{\phi}$ are in the fundamental representation of SU(4) and $\eta, \widetilde{\eta}$ are in the fundamental representation of SU(2,2) and $\phi \in L_S$, $\widetilde{\phi} \in L_S^\perp$, $\eta \in L_A$, $\widetilde{\eta} \in L_A^\perp$. Let us introduce an orthonormal basis in $\rho_{\mathrm{SU}(2,2)} \oplus \rho^*_{\mathrm{SU}(4)}$, according to the decomposition $L_A \oplus L_A^\perp \oplus L_S^* \oplus (L_S^\perp)^*$. In this basis, the antihermitean matrix

$$\begin{bmatrix} 0 & 0 & 0 & \eta \otimes \widetilde{\phi} \\ 0 & 0 & \widetilde{\eta} \otimes \phi & 0 \\ 0 & -\phi^* \otimes \widetilde{\eta}^* & 0 & \\ -\widetilde{\phi}^* \otimes \eta^* & 0 & 0 & 0 \end{bmatrix} \tag{3.25}$$

defines an infinitesimal variation of the plane $L_A \oplus L_S^*$ in the 8-dimensional space $\rho_{\mathrm{SU}(2,2)} \oplus \rho^*_{\mathrm{SU}(4)}$, such that the variation of L_A goes outside $\rho_{\mathrm{SU}(2,2)}$ into $\rho^*_{\mathrm{SU}(4)}$, and the variation of L_S^* goes into $\rho_{\mathrm{SU}(2,2)}$.

We have the following picture. The null-surface is a collection of the light rays. Each light ray defines an equator in AdS_5 or equivalently a 2-plane $L_A \subset \rho_{\mathrm{SU}(2,2)}$, and an equator in S^5 or equivalently a 2-plane $L_S^* \subset \rho^*_{\mathrm{SU}(4)}$. Turning on θ^I corresponds to the deformation of $L_A \oplus L_S^*$ inside $\rho_{\mathrm{SU}(2,2)} \oplus \rho^*_{\mathrm{SU}(4)}$, so that L_A is not entirely inside of $\rho_{\mathrm{SU}(2,2)}$ and L_S^* is not entirely in $\rho^*_{\mathrm{SU}(4)}$. In other words, while a "purely bosonic" null ray parametrizes a pair of 2-planes

$$(L_A \subset \rho_{\mathrm{SU}(2,2)},\; L_S^* \subset \rho^*_{\mathrm{SU}(4)})$$

a null ray with θ^I parametrizes a 4-plane

$$L_A \oplus L_S^* \subset \rho_{\mathrm{SU}(2,2)} \oplus \rho^*_{\mathrm{SU}(4)}$$

JHEP09(2004)068

without the constraints that L_A belongs to $\rho_{\mathrm{SU}(2,2)}$ or L_S^* belongs to $\rho^*_{\mathrm{SU}(4)}$. But we also have to take into account that θ^I are odd coordinates. This can be done by declaring $\rho_{\mathrm{SU}(2,2)}$ an "even" space and $\rho^*_{\mathrm{SU}(4)}$ an "odd" space. The total space is now $\rho_{\mathrm{SU}(2,2)} \oplus \Pi\rho^*_{\mathrm{SU}(4)}$ where Π means that the vector space is considered odd. And our 4-plane $L_A \oplus L_S^*$ is actually $L_A \oplus \Pi L_S^*$. We are embedding the complex space of dimension $(2|2)$ in the complex space of dimension $(4|4)$:

$$L_A \oplus \Pi L_S^* \quad \subset \quad \rho_{\mathrm{SU}(2,2)} \oplus \Pi\rho^*_{\mathrm{SU}(4)} \,. \tag{3.26}$$

This means that turning on the fermionic degrees of freedom replaces the product of two ordinary grassmanians with the super-grassmanian:

$$\frac{\mathrm{SU}(2,2)}{S(\mathrm{U}(2)\times\mathrm{U}(2))} \times \frac{\mathrm{SU}(4)}{S(\mathrm{U}(2)\times\mathrm{U}(2))} \rightarrow \frac{\mathrm{PSU}(2,2|4)}{\mathrm{PSU}(2|2)\times\mathrm{PSU}(2|2)\times\mathrm{U}(1)^2}\,. \tag{3.27}$$

3.3 The hamiltonian system

The space of parametrized null surfaces comes with a natural hamiltonian dynamics. The hamiltonian as a functional of a contour in the coset space is equal to the action of the particle, which has this contour as a trajectory. The symplectic form is the Kahler form on the coset integrated over the contour. (It is obtained as a limit of the natural symplectic form on the space of the classical string worldsheets.) With the fermionic degrees of freedom turned on, the hamiltonian is the action of the particle moving on the super-grassmanian (3.27). The metric on this space is fixed by the supersymmetry, and the symplectic form follows from the symplectic form of the string worldsheet theory. The action of the string worldsheet theory is:

$$\frac{1}{\epsilon}\int d\tau d\sigma \left\{ (\partial_\tau x)^2 - \epsilon^2(\partial_\sigma x)^2 + (\overline{\theta^1}, \Gamma^A \widehat{\partial_+ x}\; D_-\theta^1) + (\overline{\theta^2}, \Gamma^A \widehat{\partial_- x}\; D_+\theta^2) \right\}. \tag{3.28}$$

To compute the symplectic form at the leading order in $\frac{1}{\epsilon}$ we substitute for x and θ a parametrized null-surface. We get for the symplectic form:[2]

$$\begin{aligned}\omega = {} & \frac{1}{\epsilon} d\phi \wedge dE + \\ & + \frac{1}{\epsilon}\int d\sigma \; \left[(\overline{dY} \wedge dY) + (\overline{dZ}\wedge dZ) + \left(d\left(\overline{\Psi_{++}}\,[\widehat{\overline{Y}}, \widehat{Y}]\right) \wedge d\Psi_{++}\right)\right],\end{aligned} \tag{3.29}$$

where ϕ is the relative phase of Y and Z, and E is its conjugate variable (see [12]). This expression is written in the gauge $(\overline{Y}, \partial_\sigma Y) = (\overline{Z}, \partial_\sigma Z) = \mathrm{const}$ and with the condition (3.23).

A surprising feature of the hamiltonian is that it contains a fermionic bilinear with two derivatives. Therefore the evolution equation for the fermion should be of the form

$$\partial_t \Psi_{++} = \partial_\sigma^2 \Psi_{++}\,. \tag{3.30}$$

[2] The relative sign of $|dY|^2$ and $|dZ|^2$ is different from what we had in (1.1) because in this section we are using the "mostly plus" convention for the metric following [27]. With this convention $|dY|^2 = -|dY_{-1}|^2 - |dY_0|^2 + \sum_{i=1}^4 |dY_i|^2$ and $|dZ|^2 = \sum_{I=1}^6 |dZ_I|^2$.

JHEP09(2004)068

In the Metsaev-Tseytlin action the fermion derivatives entered only in the combinations of the form $\theta d\theta$. Therefore one would expect equations of motion of the form $\partial_t \Psi_{++} = \partial_\sigma \Psi_{++}$ rather than (3.30). But it turns out that after taking the average of the equation of motion over the period the first derivative $\partial_\sigma \Psi_{++}$ is replaced by the second derivative. Indeed, let us consider the equation for fermions:

$$\partial_\tau \Psi_{++} - \epsilon\, \Gamma^A \Gamma^S (\partial_\sigma \Psi_{++})^* = 0\,. \tag{3.31}$$

This equations looks suspicious, because the τ-derivative of Ψ_{++} is of the order ϵ rather than ϵ^2. The slow evolution should be on the time scale $\Delta\tau \sim \frac{1}{\epsilon^2}$, not $\frac{1}{\epsilon}$. But remember that we have to take the average over the period. If we neglect the terms of the order ϵ^2 and higher, we will get

$$\Psi(\tau + 2\pi, \sigma) - \Psi(\tau, \sigma) = -2\pi\epsilon\, \langle \Gamma^A \Gamma^S \rangle (\partial_\sigma \Psi_{++})^*\,. \tag{3.32}$$

Here $\langle \Gamma^A \Gamma^S \rangle$ means the average of $\Gamma^A \Gamma^S$ over the period:

$$\langle \Gamma^A \Gamma^S \rangle = \frac{1}{2}\left(\Gamma^A \Gamma^S + \widehat{\partial_\tau x_A} \widehat{\partial_\tau x_S} \right). \tag{3.33}$$

But the image of this operator is a kappa-symmetry (3.21):

$$\left(\Gamma^A \Gamma^S + \widehat{\partial_\tau x_A} \widehat{\partial_\tau x_S} \right) (\partial_\sigma \Psi_{++})^* = \left(\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S \right) \widehat{\partial_\tau x_A} \Gamma^S \, (\partial_\sigma \Psi_{++})^*$$

Therefore the slow evolution of Ψ_{++} in the order ϵ is trivial. Let us compute the order ϵ^2. To simplify the calculations, we will assume that $\partial_\sigma x = 0$. The variation of Ψ over the period is:

$$\Psi(\tau + 2\pi, \sigma) - \Psi(\tau, \sigma) = -\epsilon^2 \int_0^{2\pi} d\tau' \int_0^{\tau} d\tau'\, \Gamma^A(\tau) \Gamma^S(\tau) \Gamma^A(\tau') \Gamma^S(\tau') \;\; \partial_\sigma^2 \Psi_{++}\,. \tag{3.34}$$

Notice that

$$\begin{aligned} \Gamma^A(\tau) \Gamma^A(\tau') &= -\cos(\tau - \tau') + \sin(\tau - \tau')\, \widehat{\partial_\tau x_A} \Gamma^A \\ \Gamma^S(\tau) \Gamma^S(\tau') &= \cos(\tau - \tau') + \sin(\tau - \tau')\, \widehat{\partial_\tau x_S} \Gamma^S\,. \end{aligned} \tag{3.35}$$

After taking the integrals we get:

$$\begin{aligned} &\Psi_{++}(\tau + 2\pi, \sigma) - \Psi_{++}(\tau, \sigma) = \\ &\quad = \epsilon^2 \left[\pi^2 (1 - \widehat{\partial_\tau x_A} \Gamma^A \widehat{\partial_\tau x_S} \Gamma^S) - \frac{\pi}{2} (\widehat{\partial_\tau x_A} \Gamma^A - \widehat{\partial_\tau x_S} \Gamma^S) \right] \partial_\sigma^2 \Psi_{++}\,. \end{aligned}$$

The first term in the square brackets is again a kappa-symmetry (3.21). The second term is an operator constant on the light ray, multiplying the second derivative of Ψ_{++}. Now we can introduce the slow time $t = \epsilon^2 \tau$ and write down the equation for the slow evolution:

$$\partial_t \Psi_{++} = \frac{1}{2} \widehat{\partial_\tau x_A} \Gamma^A \, \partial_\sigma^2 \Psi_{++}\,. \tag{3.36}$$

JHEP09(2004)068

The structure of this equation implies that the hamiltonian of the slow evolution is of the form

$$H = \int d\sigma \ \left[(\partial_\sigma \overline{Y}, \partial_\sigma Y) + (\partial_\sigma \overline{Z}, \partial_\sigma Z) + (\partial_\sigma \overline{\Psi}_{++}, \partial_\sigma \Psi_{++})\right] . \tag{3.37}$$

We should stress that this expression for the hamiltonian is valid only in the quadratic order in Ψ_{++}. The difference in the structure of the fermionic term in (3.29) and (3.37) should be related to the action of the complex structure on Ψ_{++}.

The terms of the higher order in θ should be fixed by the supersymmetry. In fact, the quadratic terms are also fixed by the superconformal symmetry and locality. Still, we think it is a nontrivial fact that the moduli space of null-surfaces in AdS times a sphere has a natural structure of the hamiltonian system, and that the degrees of freedom are roughly the same as needed to parametrize the single trace operators composed of the large number of elementary fields. And if it is possible to understand the higher loop dynamics along the lines of [8]–[10], it is very unlikely that it would be also fixed by the superconformal symmetry.

Acknowledgments

I would like to thank I. Bars, J. Gomis, A. Kapustin, J.H. Schwarz and E. Witten for discussions. This research was supported by the Sherman Fairchild Fellowship and in part by the RFBR Grant No. 03-02-17373 and in part by the Russian Grant for the support of the scientific schools No. 00-15-96557.

References

[1] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226].

[2] A.A. Tseytlin, *Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond*, *Int. J. Mod. Phys.* **A 18** (2003) 981 [hep-th/0209116].

[3] J.G. Russo, *Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 038 [hep-th/0205244].

[4] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, hep-th/0311139.

[5] D. Mateos, T. Mateos and P.K. Townsend, *Supersymmetry of tensionless rotating strings in $AdS_5 \times S^5$ and nearly-BPS operators*, *J. High Energy Phys.* **12** (2003) 017 [hep-th/0309114]; *More on supersymmetric tensionless rotating strings in $AdS_5 \times S^5$*, hep-th/0401058.

[6] M. Kruczenski, *Spin chains and string theory*, hep-th/0311203.

[7] N. Beisert, *The* su(2|3) *dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252].

[8] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, *J. High Energy Phys.* **03** (2004) 004 [hep-th/0310182]; *Two-loop commuting charges and the string/gauge duality*, hep-th/0403077.

[9] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *J. High Energy Phys.* **05** (2004) 024 [hep-th/0402207].

JHEP09(2004)068

[10] M. Kruczenski, A.V. Ryzhov and A.A. Tseytlin, *Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains*, *Nucl. Phys.* **B 692** (2004) 3 [hep-th/0403120].

[11] A. Mikhailov, *Speeding strings*, *J. High Energy Phys.* **12** (2003) 058 [hep-th/0311019].

[12] A. Mikhailov, *Slow evolution of nearly-degenerate extremal surfaces*, hep-th/0402067.

[13] R. Hernandez and E. Lopez, *The* SU(3) *spin chain sigma model and string theory*, *J. High Energy Phys.* **04** (2004) 052 [hep-th/0403139].

[14] J. Stefański, B. and A.A. Tseytlin, *Large spin limits of AdS/CFT and generalized Landau-lifshitz equations*, *J. High Energy Phys.* **05** (2004) 042 [hep-th/0404133].

[15] E. Ivanov, L. Mezincescu and P.K. Townsend, *Fuzzy $CP^{(n|m)}$ as a quantum superspace*, hep-th/0311159; *A super-flag Landau model*, hep-th/0404108.

[16] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **03** (2003) 013 [hep-th/0212208].

[17] N. Beisert, C. Kristjansen, M. Staudacher, *The dilatation operator of conformal $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131, [hep-th/0306139];
N. Beisert, *The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015];
N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[18] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. and Quant. Grav.* **1** (1984) 469, erratum *ibid.* **2** (1985) 127.

[19] J. Lukierski and A. Nowicki, *General superspaces from supertwistors*, *Phys. Lett.* **B 211** (1988) 276.

[20] P.S. Howe and P.C. West, *Superconformal invariants and extended supersymmetry*, *Phys. Lett.* **B 400** (1997) 307 [hep-th/9611075];
G.G. Hartwell and P.S. Howe, (n, p, q) *harmonic superspace*, *Int. J. Mod. Phys.* **A 10** (1995) 3901 [hep-th/9412147];
P.J. Heslop and P.S. Howe, *Aspects of $N = 4$ SYM*, *J. High Energy Phys.* **01** (2004) 058 [hep-th/0307210].

[21] G.W. Gibbons, *Holography and the future tube*, *Class. and Quant. Grav.* **17** (2000) 1071 [hep-th/9911027].

[22] A.M. Perelomov, *Coherent states for arbitrary Lie groups*, *Commun. Math. Phys.* **26** (1972) 222;
M.I. Monastyrsky, A.M. Perelomov, *Coherent states and symmetric spaces. II*, *Annales de l'Institut Henri Poincaré*, Section A, vol. XXIII, n. 1, 1975.

[23] V.S. Vladimirov, A.G. Sergeev, *Compactification of Minkowski space and complex analysis in the future tube*, *Ann. Polon. Math.* **46** (1985) 439.

[24] A.M. El Gradechi and S. De Bièvre, *Phase space quantum mechanics and the ant-De-Sitter spacetime and its Poincaré contraction*, *Ann. Phys. (NY)* **235** (1994) 1 [hep-th/9219133].

JHEP09(2004)068

[25] I. Bars, *Hidden symmetries, $AdS_D \times S^n$ and the lifting of one- time-physics to two-time-physics*, *Phys. Rev.* **D 59** (1999) 045019 [hep-th/9810025]; *2T physics formulation of superconformal dynamics relating to twistors and supertwistors*, *Phys. Lett.* **B 483** (2000) 248 [hep-th/0004090]; *2T physics 2001*, hep-th/0106021.

[26] Y.S. Duan, Y. Jiang and T. Xu, *The topological structure of the vortices in the $O(n)$ symmetric TDGL model*, *Phys. Lett.* **A 252** (1999) 307 [hep-th/9809014].

[27] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[28] C. Bär, *Real Killing spinors and holonomy*, *Commun. Math. Phys.* **154** (1993) 509; A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *J. High Energy Phys.* **11** (2000) 027 [hep-th/0010206].